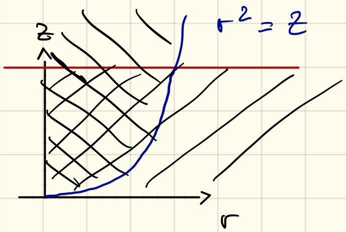
Série 7 Ex 3

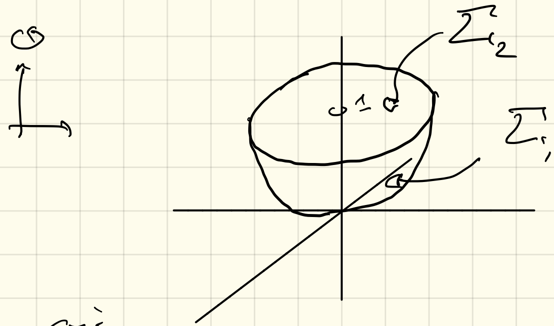
$\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 1\}$

Coordonnées cylindriques :

$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ r \geq 0,\ \theta \in [0,2\pi],\ z \in \mathbb{R}$

$x^2+y^2 \leq z \leq 1 \iff r^2 \leq z \leq 1$





$\partial\Omega = \Sigma_1 \cup \Sigma_2$ où

Σ_1 et la partie parabollique, Σ_2 est le "couvercle"

Ex 4

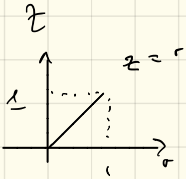
$$\Sigma_1 = \{(x,y,z)\in\mathbb{R}^3 : z^2 = x^2+y^2 \text{ et } 0\le z\le 1\}$$

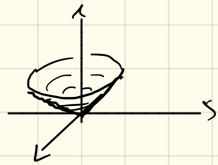
coordonnées cylindriques :

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z,\ r\geq 0,\ \theta\in[0,2\pi],\ z\in\mathbb{R}.$$

$$z^2 = x^2+y^2 \text{ et } 0\le z\le 1 \iff z^2 = r^2,\ 0\le z\le 1$$

$$\underset{r\geq 0}{\overset{z\geq 0}{\iff}} \quad z = r,\quad 0\le z\le 1$$



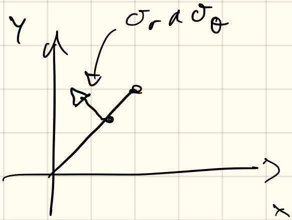


$$\sigma(r,\theta) = (r\cos\theta, r\sin\theta, r),\quad r\in[0,1],\ \theta\in[0,2\pi]$$

$$\sigma_r = (\cos\theta, \sin\theta, 1),\quad \sigma_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\sigma_r \wedge \sigma_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos & 0 \end{vmatrix} = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix}$$

$\boxed{\theta = 0}$



la normale pointe dans la bonne direction

$$\sigma_r \wedge \sigma_\theta (r, 0) = (-r, 0, r)$$

$$\iint_{\Sigma_1} F\,ds = \int_0^{2\pi}\int_0^1 \langle F(r\cos\theta, r\sin\theta, r) ; (-r\cos\theta, -r\sin\theta, r)\rangle\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \langle (r^2\cos^2\theta, r^2\sin^2\theta, r^2) ; (-r\cos\theta, -r\sin\theta ; r)\rangle\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r^3\cos^3\theta - r^3\sin^3\theta + r^3\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r^3\cos\theta(1-\sin^2\theta) - r^3\sin\theta(1-\cos^2\theta) + r^3\, dr\,d\theta$$

$$f(x) = x - \frac{1}{3}x^3 \Rightarrow \cos\theta(1-\sin^2\theta) = (\sin\theta)' f'(\sin\theta)$$

$$\Rightarrow \sin\theta(1-\cos^2\theta) = -(\cos\theta)' f'(\cos\theta)$$

$$\Rightarrow \int \cos\theta(1-\sin^2\theta)\,d\theta = f(\sin\theta)$$

$$\Rightarrow \int \sin\theta(1-\cos^2\theta)\,d\theta = -f(\cos\theta)$$

ou utiliser Série 1 ex 7

$$= \int_0^1 2\pi r^3 + r^3\left[-\left(\sin\theta - \frac{1}{3}\sin^3\theta\right) + \left(\cos\theta - \frac{1}{3}\cos^3\theta\right)\right]_{\theta=0}^{\theta=2\pi} dr$$

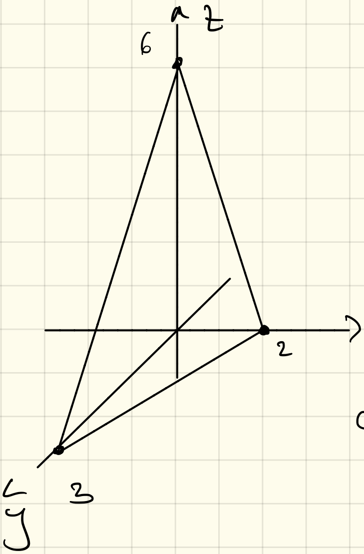
$$= \int_0^1 2\pi r^3 + r^3\left(1 - \frac{1}{3} - 1 + \frac{1}{3}\right) dr = 2\pi\int_0^1 r^3 = \frac{\pi}{2}$$

<u>Ex 5</u> $\Sigma_1 = \{ (x,y,z) \in \mathbb{R}^3 : z = 6 - 3x - 2y, \quad x,y,z \geq 0 \}$.

$(0,0,z) \in \Sigma_1 \iff z = 6$

$(0,y,0) \in \Sigma_1 \iff y = 3$

$(x,0,0) \in \Sigma_1 \iff x = 2$



notons que :

$6 - 3x - 2y = z \geq 0$

$\iff 3x + 2y \leq 6$

$\iff y \leq 3 - \frac{3}{2}x$

$\sigma(x,y) = (x, y, 6 - 3x - 2y)$,

$x \in [0,2]$, $y \in [0, 3 - \frac{3}{2}x]$.

Ex 7 $\Omega = \{(x,y,z) \in \mathbb{R}^3 : b^2(x^2+y^2) < a^2z^2, 0<z<b\}$

coordonnées cylindriques

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $r \geq 0$, $\theta \in [0, 2\pi]$, $z \in \mathbb{R}$.

$b^2(x^2+y^2) < a^2z^2$, $0<z<b$

Jacobien $= r$

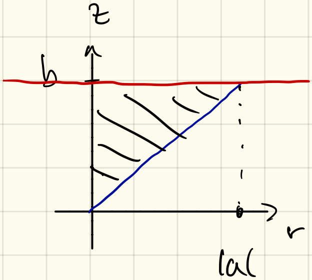
$\iff$ $b^2r^2 < a^2z^2$, $0<z<b$

$b>0$

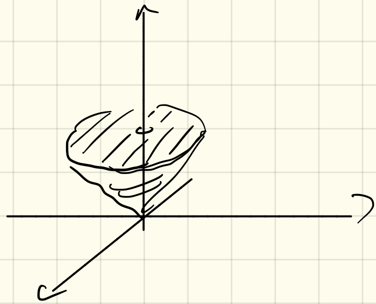
$\iff$ $0 \leq r < \frac{|a|z}{b}$, $0<z<b$.

$z>0$

$r \geq 0$



$r = \frac{|a|}{b} z$



$x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $\theta \in [0, 2\pi]$, $z \in [0,b]$,

$r \in [0, \frac{|a|}{b} z]$

$$\operatorname{div} F(x,y,z) = \frac{\partial F_1}{\partial x}(x,y,z) + \frac{\partial F_2}{\partial y}(x,y,z) + \frac{\partial F_3}{\partial z}(x,y,z)$$

$$= 2x + 2y + 2z = 2(x+y+z).$$

$$\iiint_{\Omega} \operatorname{div} F \, dx\,dy\,dz = \int_0^{2\pi}\int_0^{b}\int_0^{\frac{|a|}{b}z} 2(r\cos\theta + r\sin\theta + z)\, r\, dr\, dz\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^{b}\int_0^{\frac{|a|}{b}z} r^2\cos\theta + r^2\sin\theta + rz \, dr\, dz\, d\theta$$

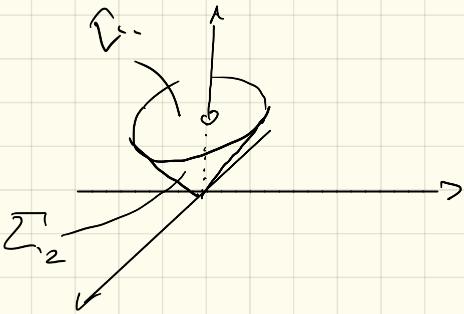
$$= 2\int_0^{b}\int_0^{\frac{|a|}{b}z} r^2 \underbrace{\int_0^{2\pi} \cos\theta + \sin\theta \, d\theta}_{=0} \, dr\, dz$$

$$+ 4\pi \int_0^{b} \left[\frac{1}{2} r^2 z\right]_{r=0}^{r=\frac{|a|}{b}z} dz$$

$$= 2\pi \int_0^{b} \frac{a^2}{b^2} z^3 \, dz = 2\pi \left[\frac{1}{4}\frac{a^2}{b^2} z^4\right]_{z=0}^{z=b} = \frac{\pi}{2} a^2 b^2$$

$\partial\Omega = \Sigma_1 \cup \Sigma_2$ avec Σ_1 le couvercle

et Σ_2 le cône,



$$\Sigma_1 = \{(x,y,z)\in\mathbb{R}^3 : b^2(x^2+y^2) \le a^2 z^2, \quad z = b\}$$

$$= \{(x,y,b)\in\mathbb{R}^3 : x^2+y^2 \le a^2\}$$

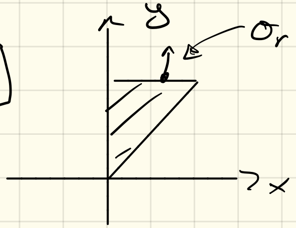
$$\sigma(r,\theta) = (r\cos\theta, r\sin\theta, b), \qquad r\in[0,|a|].$$

$$\sigma_r = (\cos\theta, \sin\theta, 0)$$

$$\sigma_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\sigma_r \wedge \sigma_\theta = \begin{vmatrix} e_1 & e_2 & e_3 \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

$\boxed{\theta = 0}$



$\sigma_r \wedge \sigma_\theta$ est bien extérieure

$$\iint_{\Sigma_1} \langle F; \nu \rangle ds = \int_0^{2\pi} \int_0^{|a|} \langle (r^2\cos^2\theta, r^2\sin^2\theta, b^2); (0,0,r) \rangle dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{|a|} b^2 r \, dr d\theta = \pi b^2 a^2$$

$$\Sigma_2 = \{ (x,y,z) \in \mathbb{R}^3 : b^2(x^2+y^2) = a^2 z^2 , 0 < z < b \}$$

$$= \{ (x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} = \frac{|a|}{b} z , 0 < z < b \}$$

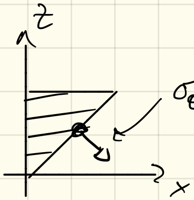
$$\sigma(\theta, z) = \left(\frac{|a|}{b} z\cos\theta, \frac{|a|}{b} z \sin\theta, z \right), \quad \theta \in [0, 2\pi], \; z \in [0,b]$$

$$\sigma_\theta = \left(-\frac{|a|}{b} z\sin\theta, \frac{|a|}{b} z\cos\theta, 0 \right)$$

$$\sigma_z = \left(\frac{|a|}{b} \cos\theta, \frac{|a|}{b} \sin\theta, 1 \right)$$

$$\sigma_\theta \wedge \sigma_z = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\frac{|a|}{b} z \sin\theta & \frac{|a|}{b} z \cos\theta & 0 \\ \frac{|a|}{b}\cos\theta & \frac{|a|}{b}\sin\theta & 1 \end{vmatrix} = \begin{pmatrix} \frac{|a|}{b} z\cos\theta \\ -\frac{|a|}{b} z \sin\theta \\ -\frac{a^2}{b^2} z \end{pmatrix}$$

$\boxed{\theta = 0}$



$\sigma_\theta \wedge \sigma_z$ est bien extérieure !

$$\iint_{\Sigma_2} \langle F ; \nu \rangle ds = \int_0^{2\pi}\int_0^b \left\langle \left(\frac{a^2}{b^2} z^2 \cos^2\theta, \frac{a^2}{b^2} z^2 \sin^2\theta, z^2 \right); \left(\frac{|a|}{b} z\cos\theta, -\frac{|a|}{b} z \sin\theta, -\frac{a^2}{b^2} z \right) \right\rangle$$

$$= \int_0^{2\pi}\int_0^b \frac{|a|^3}{b^3} z^3 \cos^3\theta - \frac{|a|^3}{b^3} \sin^3\theta - \frac{a^2}{b^2} z^3 \, dz\, d\theta$$

$$= \int_0^b \frac{|a|^3}{b^3} z^3 \underbrace{\int_0^{2\pi} \cos^3\theta \, d\theta}_{=0} - \frac{|a|^3}{b^3} z^3 \underbrace{\int_0^{2\pi} \sin^3\theta \, d\theta}_{=0}$$

$$-2\pi \frac{a^2}{b^2} z^3 dz$$

$$= -2\pi \frac{a^2}{b^2} \left[\frac{1}{4} z^4\right]_{z=0}^{z=b} = -\frac{\pi}{2} a^2 b^2$$

Enfin,

$$\iint_{\partial\Omega} \langle F; \nu \rangle ds = \iint_{\Sigma_1} \langle F; \nu \rangle ds + \iint_{\Sigma_2} \langle F; \nu \rangle ds$$

$$= \pi a^2 b^2 - \frac{\pi}{2} a^2 b^2 = \frac{\pi}{2} a^2 b^2 = \iiint_{\Omega} \operatorname{div} F \, dx dy dz$$